Technical Notes

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Turbulent Compressible Boundary Layer with Heat-Transfer Effects on Conditions at the Sublayer

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IN 1950, the author published turbulent compressible skin-friction equations limited to zero heat transfer.[1] The theory was based on von Kármán's differential equation for the boundary-layer velocity distribution. A theory by Van Driest, published in 1951, was based on Prandtl's differential equation and included heat transfer.[2] In 1956, Van Driest used von Kármán's equation and revised his results, referred to in the literature as Van Driest II.[3] The Van Driest II results and other theories have been compared with experiment by several authors. For example, Hopkins et al.[4] compared theories with direct friction measurements in air at a freestream Mach number M_I of 5.9 to 7.8 and a wall to recovery temperature ratio T_w/T_r of 0.3 to 0.5. The theories included Van Driest II, Coles,[5] Sommer and Short,[6] Spalding and Chi,[7] and finite-difference calculations described in Ref. 4. The Van Driest II theory gave the best results and checked experiment within ±10%.

Although Van Driest II gives good agreement with experiment for comparisons made to date, the theory assumes constant conditions at the laminar sublayer edge. Heat transfer affects these conditions, and a theory for flow over a flat plate presented here accounts for these effects. Calculations are made to determine their importance to skin-friction predictions over a wide range of Mach numbers and heat-transfer rates. For air at Mach number 15, the predictions are 15% lower to 15% higher than Van Driest, depending on heat-transfer rate. Recent limited friction measurements in helium at $M_I = 11.3$ have been published by Watson for $T_w/T_r = 0.4$ to 1.0.[8] For these conditions, the present theory and Van Driest II differ by as much as 25%, the data tending to validate the present theory.

Theory

The von Kármán differential equation for the boundary-layer velocity distribution in incompressible flow is[9]

$$\sqrt{\frac{\tau}{\rho}} = -\kappa\left[\left(\frac{du}{dy}\right)^2 \Big/ \frac{d^2u}{dy^2}\right] \tag{1}$$

Compressible theory assumes Eq. (1) holds when the constant density restriction is removed. The shear stress τ is assumed constant and equal to the value at the surface. The "universal" constant κ is assumed to be unaffected by compressibility and equal to 0.4, the value from incompressible flow. After writing

$$\tau/\rho_w = u_\tau \tag{2a}$$

$$\phi = u/u_\tau \tag{2b}$$

$$\eta = \rho_w u_\tau y/\mu_w \tag{2c}$$

$$\rho/\rho_w = T_w/T \tag{2d}$$

Eq. (1) becomes

$$\frac{d^2\phi}{d\eta^2}\Big/\left(\frac{d\phi}{d\eta}\right)^2 = \frac{d\log(d\phi/d\eta)}{d\phi} = -\kappa\sqrt{\frac{T_w}{T}} \tag{3}$$

The author has given a relation between temperature and velocity[10] which can be written

$$\frac{T}{T_w} = 1 + \left(\frac{T_r}{T_w} - 1\right)\Lambda^{-1/n}\frac{\phi}{\phi_I} - \sigma_r\frac{T_r}{T_w}\Lambda^{-2/n}\left(\frac{\phi}{\phi_I}\right)^2 \tag{4}$$

Equation (4) gives good agreement with experiment for turbulent boundary layers on flat plates. The exponent n is about 7 and appears in the familiar approximation for velocity distribution, $u/u_I = (y/\delta)^{1/n}$ and Λ is the ratio of thermal to velocity boundary-layer thickness. The parameter σ_r is given by M_I and recovery factor r as

$$\sigma_r = \left(r\frac{\gamma-1}{2}M_I^2\right)\Big/\left(1 + r\frac{\gamma-1}{2}M_I^2\right)$$

Combining Eqs. (3) and (4) and integrating

$$\frac{d\eta}{d\phi} = \frac{1}{f}\exp[\kappa\beta\phi_I/\sqrt{\sigma_r T_r/T_w}] \tag{5}$$

where

$$\beta = \Lambda^{1/n}\left\{\sin^{-1}\left|\frac{2\Lambda^{-1/n}\sigma_r\frac{\phi}{\phi_I} - \left(1 - \frac{T_w}{T_r}\right)}{\sqrt{4\sigma_r\frac{T_w}{T_r} + \left(1 - \frac{T_w}{T_r}\right)^2}}\right| - \sin^{-1}\left|\frac{2\Lambda^{-1/n}\sigma_r\frac{S}{\phi_I} - \left(1 - \frac{T_w}{T_r}\right)}{\sqrt{4\sigma_r\frac{T_w}{T_r} + \left(1 - \frac{T_w}{T_r}\right)^2}}\right|\right\} \tag{6}$$

To obtain Eq. (5), the constant of integration was evaluated at the edge of the laminar sublayer by putting $d\phi/d\eta = f$ when $\phi = S$.

The values of S and f are affected by heat transfer. Harkness[11] gives the following expression for S:

$$S = 11.5 + 6.6[1 - (T_w/T_r)] \tag{7}$$

The preceding result was obtained by fitting experimental data for S at $M_I = 2.4$ to 8.2 and $T_w/T_r = 0.5$ to 1.7. The experimental values for S are given by the intersection between $\phi = \eta$ and an experimental curve for the turbulent portion of the boundary layer.

The effect of heat transfer on f can be determined from Prandtl's equation for the turbulent velocity distribution, $\sqrt{\tau/\rho} = \kappa y du/dy$. This equation holds near the wall and can be

Received Oct. 3, 1978; revision received Jan. 24, 1979. This paper is declared a work of the U.S. Government and therefore is in the public domain.

Index category: Boundary Layers and Convective Heat Transfer—Turbulent.

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written $\kappa\eta \mathrm{d}\phi/\mathrm{d}\eta = \sqrt{T/T_w}$. At $\phi = \eta = S$, $\mathrm{d}\phi/\mathrm{d}\eta = f$ and $T = T_s$. Therefore, f is given by

$$f = 1/\kappa S\sqrt{T_s/T_w} \tag{8}$$

where T_s/T_w is obtained by substituting $\phi = S$ in Eq. (4).

The skin-friction equations can now be derived from the definition of boundary-layer momentum thickness θ and the momentum equation written in the form $C_f = 2\mathrm{d}\theta/\mathrm{d}x$. They can be combined to yield

$$\mathrm{d}R_w = \frac{2}{C_{f_w}}\mathrm{d}\left\{\phi_1^2 \int_0^1 \frac{(\phi/\phi_1)[1-(\phi/\phi_1)]}{T/T_w}\frac{\mathrm{d}\eta}{\mathrm{d}\phi}\mathrm{d}\left(\frac{\phi}{\phi_1}\right)\right\} \tag{9}$$

where the Reynolds number R_w and local skin-friction coefficient C_{f_w} are based on wall conditions. Making use of Eqs. (4) and (5), Eq. (9) can be solved by a procedure similar to that used in Ref. 2. With f given by Eq. (8) and noting that $C_{f_w} = 2/\phi_1^2$, the result is

$$\beta_1 \Big/ \sqrt{\sigma_r \frac{T_r}{T_w} C_{f_w}} = \frac{1}{\kappa\sqrt{2}}\log\frac{\kappa e^{\kappa S_i}}{2S_i} - \frac{S_i}{\sqrt{2}} + \frac{1}{\kappa\sqrt{2}}\log\left(\frac{S_i}{S}\sqrt{\frac{T_s}{T_w}}C_{f_w}R_w\right) \tag{10}$$

where $S_i = 11.5$, the value of S from incompressible flow and also given by Eq. (7) when $T_w/T_r = 1.0$. To obtain β_1, put $\phi = \phi_1$ in Eq. (6). For the incompressible case, Eq. (10) reduces to

$$\frac{1}{\sqrt{C_{f_i}}} = \frac{1}{\kappa\sqrt{2}}\log\frac{\kappa e^{\kappa S_i}}{2S_i} + \frac{1}{\kappa\sqrt{2}}\log(C_{f_i}R)$$

This is von Kármán's equation. He obtained values of the constants from friction experiments with the familiar result

$$1/\sqrt{C_{f_i}} = 1.7 + 4.15\log_{10}(C_{f_i}R) \tag{11}$$

With $S_i = 11.5$, the preceding constants, and converting to freestream conditions, Eq. (10) becomes

$$\beta_1 \Big/ \sqrt{\sigma_r \frac{T_r}{T_1} C_f} = -6.43 + 4.15\log_{10}\left\{\frac{S_i}{S}\frac{\mu_1}{\mu_w}\sqrt{\frac{T_s}{T_w}}C_f R\right\} \tag{12}$$

According to Schoenherr,[12] the incompressible mean skin-friction coefficient can be written in the same form as the local coefficient with different constants. The same result holds for compressible flow with the following result for mean skin-friction coefficient C_F

$$0.242\beta_1 \Big/ \sqrt{\sigma_r \frac{T_r}{T_1} C_F} = -1.97 + \log_{10}\left\{\frac{S_i}{S}\frac{\mu_1}{\mu_w}\sqrt{\frac{T_s}{T_w}}C_F R\right\} \tag{13}$$

The parameter $\Lambda^{1/n}$ appears in the expression for β_1 and also for T_s/T_w. The values given in Ref. 10 are very near 1.0, and calculations have shown that taking $\Lambda^{1/n} = 1$ has a negligible effect on skin-friction values. Therefore, from Eqs. (4) and (6), the expressions for T_s/T_w and β_1 to be used with Eq. (12) are

$$\frac{T_s}{T_w} = 1 + \left(\frac{T_r}{T_w} - 1\right)S\sqrt{\frac{T_w}{T_1}\frac{C_f}{2}} - \sigma_r\frac{T_r}{T_1}S^2\frac{C_f}{2} \tag{14}$$

$$\beta_1 = \sin^{-1}\frac{2\sigma_r - \left(1 - \frac{T_w}{T_r}\right)}{\sqrt{4\sigma_r\frac{T_w}{T_r} + \left(1 - \frac{T_w}{T_r}\right)^2}} - \sin^{-1}\frac{2\sigma_r S\sqrt{\frac{T_w}{T_1}\frac{C_f}{2}} - \left(1 - \frac{T_w}{T_r}\right)}{\sqrt{4\sigma_r\frac{T_w}{T_r} + \left(1 - \frac{T_w}{T_r}\right)^2}} \tag{15}$$

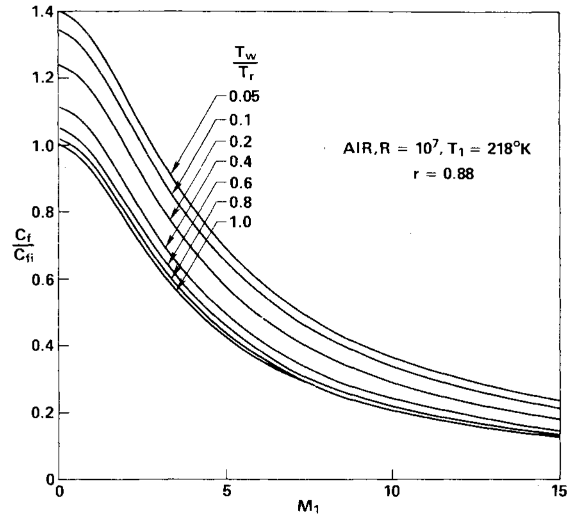

Fig. 1 Effect of Mach number and wall temperature on skin friction.

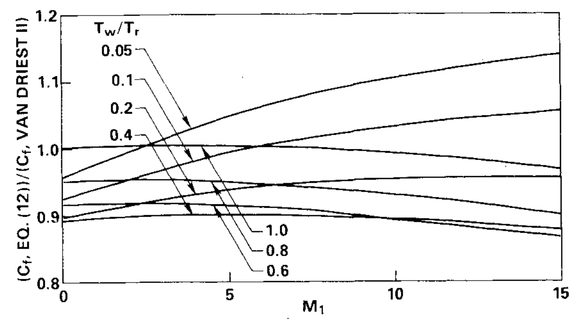

Fig. 2 Comparison of theories: air, $R = 10^7$, $T_1 = 218$ K, $r = 0.88$.

The expressions for T_s/T_w and β_1 for use with Eq. (13) are obtained from Eqs. (14) and (15) by substituting C_F for C_f.

Equation (12) has been used to compute local skin-friction coefficient for air over a wide range of Mach numbers and wall temperatures. The results, normalized by C_{f_i} from Eq. (11), are plotted on Fig. 1. The Sutherland law for air, $\mu \sim T^{3/2}/(T+110)$, T in K, was used to compute viscosity ratios.

Comparison with Van Driest and Experiment

Skin-friction coefficients in air from Eq. (12) are compared with Van Driest II results on Fig. 2. The differences between the present results and Van Driest II increase with Mach number. For the wall temperatures shown, the differences approach $\pm 15\%$ at $M_1 = 15$. The differences are due in great part to the effects of heat transfer on conditions at the sublayer edge. The Van Driest II theory assumes the conditions are unaffected by heat transfer; and at zero heat transfer, $T_w/T_r = 1.0$, the two theories give nearly the same result. It should be noted that Van Driest proposes a power law for viscosity, $\mu \sim T^m$. For the present comparison, the Sutherland law was used with both theories.

Recent direct skin-friction measurements by Watson[8] are plotted on Fig. 3. The measurements were made in helium at $M_1 = 11.3$. From measurements of the location of peak heating, associated with the transition from laminar to turbulent flow, Watson determined the virtual origin of the turbulent boundary layer to be at $R = 3.2 \times 10^7$. In plotting the data on Fig. 3, each of the test Reynolds numbers was reduced by this amount. The experimental results can now be compared with theory for 100% turbulent flow. Skin-friction coefficients from Eq. (12) and from Van Driest II are also plotted on Fig. 3. An equation for the viscosity of helium given in Ref. 8 was used for both calculations. The limited data on Fig. 3 tend to validate the present theory. For the

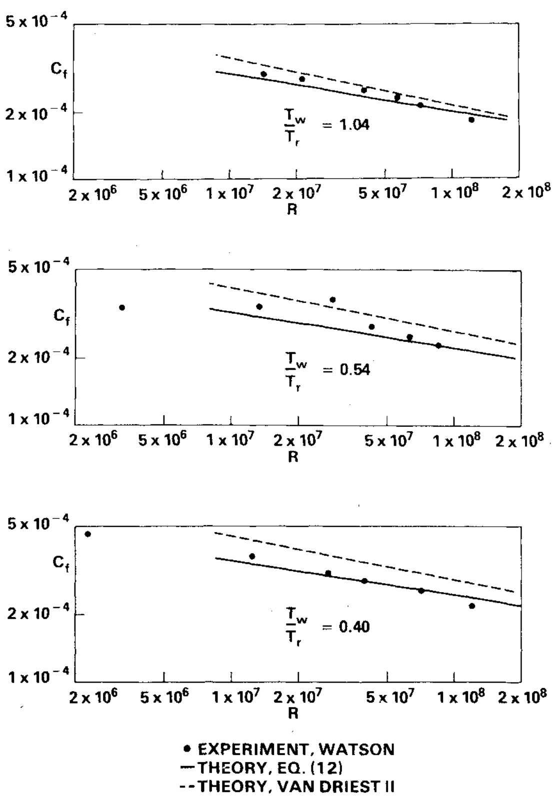

Fig. 3 Comparison of theory and experiment: helium, $M_I = 11.3$, $r = 0.883$.

lowest wall temperature ratio, $T_w/T_r = 0.40$, the agreement between Eq. (12) and experiment is excellent. In this case the data fall well below the Van Driest II result. At the higher wall temperatures, the data fall between Eq. (12) and Van Driest II. However, at the higher Reynolds numbers, where any error in the virtual origin has the least effect, Eq. (12) agrees best with the data.

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Effect of Frequency in Unsteady Transonic Flow

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Introduction

A CHARACTERISTIC of transonic unsteady flows is the potentially large phase lag between boundary motion and induced surface pressure. Moreover, net force coefficients generally exceed those in subsonic and supersonic speed regimes. These effects tend to increase the likelihood of aeroelastic instabilities, making transonic speeds most critical for aircraft flutter. In this Note the effects of frequency are systematically considered within the framework of transonic small-disturbance theory for three different configurations: airfoil pitching oscillation, trailing edge flap oscillation, and impulsive change to angle of attack. An approximate factorization method applicable to general unsteady motions is devised to study the net lift and moment coefficients for the NACA 64A010 airfoil section, at Mach 0.82 and for three reduced frequencies, namely, 0.05, 0.5, and 5.0. The results are then compared against those obtained in the low-frequency approximation.

Generally speaking, small surface motions can induce large changes in aerodynamic loading, as well as large increments in shock excursion. These considerations arise from the inherent nonlinearity of the mathematical problem. Thus, in the numerical sense, direct time integration, which does not bear the limiting restriction to "frozen" shock movements, typical of linearized approaches, must be used. In the low-frequency approximation, solutions to the transonic small-disturbance equation, along these lines, simulate well the expected flow nonlinearity, including irregular shockwave motions.[1-3] These results compare well with those obtained from the unsteady Euler equations and are in good agreement with experiment. However, the restriction to low frequencies may, in practice, be severe; more rapid oscillations, as well as unsteady gust loadings, are excluded from consideration. The essential loss is that in phase-shift information and it is remedied by retaining in the governing equations the high-frequency terms. This forms the substance of the numerical algorithm discussed in this Note. Of course, the extent to which the low-frequency approximation holds is also of fundamental interest, and comparisons are given for this purpose. This is obviously important in assessing the effect of frequency on transonic phase shifts and bears practical significance on direct aeroelastic applications.

Received March 20, 1979.

Index categories: Nonsteady Aerodynamics; Transonic Flow; Computational Methods.

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